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Note on an integral equation of viscous flow theory

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SUMMARY

The integral equation encountered by Van de Vooren and Veldman in a problem of viscous flow was recently solved by Brown by use of the Wiener–Hopf technique. In this note Brown's solution is re-derived by a different, function-theoretic method.

1. Introduction

In their analysis of the incompressible viscous flow near the leading edge of a flat plate Van de Vooren and Veldman [1] encountered the integral equation

$$f(x) = (2\pi)^{-1} \int_0^\infty \log|x-t| f(t)\,dt + x^{-\frac{1}{2}}, \tag{1}$$

where the function $f(x)$ is related to the slip velocity on the plate. An exact solution of the latter equation was recently presented by Brown [2], obtained by means of complex Fourier transforms and the Wiener–Hopf technique. Here a minor difficulty arises as the Fourier transform of the kernel $\log|x|$ does not exist. Therefore Brown introduces a suitable convergence factor which, in fact, amounts to solving the related integral equation

$$f(x) = (2\pi)^{-1} \int_0^\infty \log|x-t| e^{-\varepsilon|x-t|} f(t)\,dt + x^{-\frac{1}{2}} e^{-\varepsilon x}$$

for $\varepsilon > 0$, and then taking the limit of the solution as $\varepsilon \to 0$.

In the present note the integral equation (1) is solved by a function-theoretic method due to Heins and MacCamy [3], though essentially going back to Carleman. By use of complex Laplace transforms the solution of (1) is reduced to a Hilbert problem for a sectionally analytic function. The latter problem is treated by the standard technique described in Muskhelishvili [4]. A closed-form result is obtained for $\mathscr{L}\{f(x)\}$, the Laplace transform of $f(x)$, and on inversion Brown's solution [2] for $f(x)$ is precisely recovered. The asymptotic expansions of $f(x)$ for small and large x are obtainable directly from $\mathscr{L}\{f(x)\}$. It is found that the expansions agree with those of Brown except for a minor error in the expansion coefficient C_2 [2, eq. (6.2b)].

Two final remarks are in order. In [1] it was shown that

$$f(x) = O(x^{-\frac{1}{2}}) \text{ as } x \to 0, \quad f(x) = O(x^{-\frac{3}{4}} \log x) \text{ as } x \to \infty. \tag{2}$$

These results will serve as a priori estimates for $f(x)$ in the present analysis. Furthermore, in the sequel all expressions $\log w$ and w^{α} (w complex) are understood to denote principal values of the pertaining functions, uniquely determined by the restriction $-\pi < \arg w < \pi$. These principal values are analytic functions in the complex w-plane cut along the negative real axis.

2. Solution by a function-theoretic method

Following [3], we introduce the function

$$F(z) = \frac{1}{2\pi i}\int_0^\infty \log(t-z)f(t)dt \tag{3}$$

where z is a complex variable. In view of (2), the following properties of $F(z)$ are obvious:
(i) $F(z)$ is an analytic function in the complex z-plane cut along the positive real axis.

(ii) $$F(z) = \frac{1}{2\pi i}\int_0^\infty \log t f(t)dt + O(z^{\frac{1}{2}}) \text{ as } z \to 0, \tag{4a}$$

$$F(z) = \left[\frac{1}{2\pi i}\int_0^\infty f(t)dt\right]\log(-z) + O(z^{-\frac{1}{2}}\log z) \text{ as } z \to \infty. \tag{4b}$$

(iii) Let $F^{\pm}(x)$ denote the limits of $F(z)$ as $z \to x \pm i0$, then

$$F^{\pm}(x) = \frac{1}{2\pi i}\int_0^\infty \log|t-x|f(t)dt \mp \tfrac{1}{2}\int_0^x f(t)dt, \qquad x \geq 0. \tag{5}$$

To establish (4a), notice that $F'(z)$ is a Cauchy integral, hence, according to [4, § 29] we have $F'(z) = O(z^{-\frac{1}{2}})$ as $z \to 0$. The result in (4b) is readily found from the behaviour of

$$F(z^{-1}) = \frac{1}{2\pi i}\int_0^\infty [\log(t-z) - \log t]f(t^{-1})t^{-2}dt - \left[\frac{1}{2\pi i}\int_0^\infty f(t^{-1})t^{-2}dt\right]\log(-z)$$

near $z = 0$. In fact, $F(z) = O(z^{-\frac{1}{2}}\log z)$ as $z \to \infty$, since the integral in (4b) vanishes as found a posteriori in (32).

By means of (5) the integral equation (1) can be reduced to a functional equation between $F^{\pm}(x)$, viz.,

$$F^+(x) - F^-(x) = -2x^{\frac{1}{2}} - \tfrac{1}{2}i\int_0^x [F^+(t) + F^-(t)]dt, \qquad x \geq 0. \tag{6}$$

The latter equation is further reduced by Laplace transformation. To that purpose we introduce the complex Laplace transform

$$G(s) = \int_0^{\infty e^{i\beta}} e^{-sz/2}F(z)dz \tag{7}$$

where $\beta = -\arg s$. The factor $\frac{1}{2}$ in the exponent has been inserted for later convenience. The following properties of $G(s)$ are easily established:

(i) $G(s)$ is an analytic function in the complex s-plane cut along the positive real axis.

$$G(s) = O(s^{-1}\log s) \text{ as } s \to 0, \quad G(s) = O(s^{-1}) \text{ as } s \to \infty. \tag{8}$$
(ii)

(iii) Let $G^{\pm}(\sigma)$ denote the limits of $G(s)$ as $s \to \sigma \pm i0$, then

$$G^{\pm}(\sigma) = \int_0^{\infty} e^{-\sigma x/2} F^{\mp}(x)dx, \qquad \sigma > 0. \tag{9}$$

By Laplace transformation as in (9), the equation (6) transforms into

$$(\sigma - i)G^{+}(\sigma) - (\sigma + i)G^{-}(\sigma) = 2(2\pi)^{\frac{1}{2}}\sigma^{-\frac{1}{2}}, \qquad \sigma > 0. \tag{10}$$

Thus we have arrived at a Hilbert problem for the sectionally analytic function $G(s)$. This problem is now solved by the method of Muskhelishvili [4, Chap. 10].

We first determine a fundamental solution $X(s)$ of the corresponding homogeneous Hilbert problem. Taking logarithms we have

$$\log X^{+}(\sigma) - \log X^{-}(\sigma) = \log\frac{\sigma + i}{\sigma - i} = 2i\arctan(\sigma^{-1}), \qquad \sigma > 0, \tag{11}$$

and consequently, by Plemelj's formulae,

$$\log X(s) = \frac{1}{\pi}\int_0^{\infty}\frac{\arctan(t^{-1})}{t - s}dt. \tag{12}$$

The latter integral, to be denoted by $W(s)$, can be determined from

$$W(s) = -\tfrac{1}{2}\log(-s) + o(1) \text{ as } s \to 0, \tag{13}$$

$$W'(s) = -\frac{1}{2s} - \frac{1}{\pi}\int_0^{\infty}\frac{dt}{(1 + t^2)(t - s)} =$$
$$= -\frac{1}{2s} + \frac{1}{2}\frac{s}{1 + s^2} + \frac{1}{\pi}\frac{\log(-s)}{1 + s^2}. \tag{14}$$

Thus we find

$$X(s) = (-s)^{-\frac{1}{2}}(1 + s^2)^{\frac{1}{2}}e^{-m^*(s)} \tag{15}$$

where $m^*(s)$ is defined by

$$m^*(s) = -\frac{1}{\pi}\int_0^{s}\frac{\log(-t)}{1 + t^2}dt. \tag{16}$$

To make the definition (16) unique, it is understood that the path of integration does not cross the branch cuts along the positive real axis and along the imaginary axis from $-i\infty$ to $-i$ and from i to $i\infty$. Then $m^*(s)$ is a single-valued analytic function in the cut s-plane. The same branch cuts along the imaginary axis appear in the definition of the principal value of $(1 + s^2)^{\frac{1}{2}}$. However, it can easily be verified that the product $(1 + s^2)^{\frac{1}{2}}e^{-m^*(s)}$ is analytic at

$s = \pm i$ and continuous across the branch cuts $s = i\tau$ with $\tau < -1$ or $\tau > 1$. Therefore, the functions $(1+s^2)^{\frac{1}{2}} e^{-m^*(s)}$ and $X(s)$ are analytic in the s-plane with a single branch cut along the positive real axis. We list some further properties of $X(s)$ and $m^*(s)$:

(i) $X(s) \sim (-s)^{-\frac{1}{2}}$ as $s \to 0$, $X(s) \sim 1$ as $s \to \infty$. (17)

(ii) As $s \to \sigma \pm i0$, $\sigma > 0$, $m^*(s)$ and $X(s)$ attain the limit values

$$m^*(\sigma \pm i0) = m(\sigma) \pm i \arctan \sigma, \tag{18}$$

$$X^{\pm}(\sigma) = \frac{\sigma \pm i}{\sigma^{\frac{1}{2}}(1+\sigma^2)^{\frac{1}{2}}} e^{-m(\sigma)}, \tag{19}$$

where $m(\sigma)$ is the function introduced in [2, eq. (5.4)], viz.,

$$m(\sigma) = -\frac{1}{\pi} \int_0^{\sigma} \frac{\log t}{1+t^2} dt. \tag{20}$$

We now return to the original Hilbert problem (10). By setting $G(s) = X(s)\Phi(s)$, we find that the problem (10) reduces to

$$\Phi^+(\sigma) - \Phi^-(\sigma) = 2(2\pi)^{\frac{1}{2}} \frac{e^{m(\sigma)}}{(1+\sigma^2)^{\frac{1}{2}}}, \qquad \sigma > 0, \tag{21}$$

with the solution

$$\Phi(s) = \frac{(2\pi)^{\frac{1}{2}}}{\pi i} \int_0^{\infty} \frac{e^{m(t)}}{(1+t^2)^{\frac{1}{2}}} \frac{dt}{t-s}. \tag{22}$$

The present solution is unique because of the requirements implied by (8) and (17), on the behaviour of $\Phi(s)$ as $s \to 0$ and $s \to \infty$. To evaluate the Cauchy integral of (22) we consider the function

$$\frac{e^{m^*(z)}}{z(1+z^2)^{\frac{1}{2}}(z-s)} \tag{23}$$

which is analytic in the z-plane cut along the positive real axis and has a simple pole at $z = s$. Integrate this function around the contour formed by the circles $|z| = \delta$, $|z| = R$, and the line segments from δ to R along the upper and lower sides of the branch cut. Then by making $\delta \to 0$ and $R \to \infty$, we obtain

$$2\pi i \frac{e^{m^*(s)}}{s(1+s^2)^{\frac{1}{2}}} = 2\pi i s^{-1} + \int_0^{\infty} [e^{m^*(t+i0)} - e^{m^*(t-i0)}] \frac{dt}{t(1+t^2)^{\frac{1}{2}}(t-s)}$$

$$= 2\pi i s^{-1} + 2i \int_0^{\infty} \frac{e^{m(t)}}{(1+t^2)^{\frac{1}{2}}} \frac{dt}{t-s}. \tag{24}$$

In this manner the final solution for $G(s)$ is found to be

$$G(s) = i(2\pi)^{\frac{1}{2}}(-s)^{-\frac{3}{2}}[1 - (1 + s^2)^{\frac{1}{4}} e^{-m^*(s)}]. \tag{25}$$

By use of (5), (9), the Laplace transform of $f(x)$ can be expressed in terms of $G^{\pm}(\sigma)$, thus leading to

$$\begin{aligned} \mathscr{L}\{f(x)\} &= \int_0^\infty e^{-\sigma x/2} f(x)dx = \tfrac{1}{2}\sigma[G^+(\sigma) - G^-(\sigma)] \\ &= (2\pi)^{\frac{1}{2}}\sigma^{-\frac{1}{2}} - (2\pi)^{\frac{1}{2}} \frac{e^{-m(\sigma)}}{\sigma^{\frac{1}{2}}(1 + \sigma^2)^{\frac{1}{4}}}, \qquad \sigma > 0. \end{aligned} \tag{26}$$

Inversion of the first term $(2\pi)^{\frac{1}{2}}\sigma^{-\frac{1}{2}}$ yields a contribution $x^{-\frac{1}{2}}$ to $f(x)$. Through replacement of σ by s the final term in (26) is analytically continued into the complex s-plane cut along the negative real axis. Here the continued function $m(s)$ is defined as in (20), by

$$m(s) = -\frac{1}{\pi} \int_0^s \frac{\log t}{1 + t^2} dt \tag{27}$$

where the path of integration does not cross the branch cuts along the negative real axis and along the imaginary axis from $-i\infty$ to $-i$ and from i to $i\infty$. Notice that the branch cuts along the imaginary axis vanish for the product $(1 + s^2)^{-\frac{1}{4}} e^{-m(s)}$. Then the inverse of (26) is given by the Laplace inversion formula, viz.,

$$f(x) = x^{-\frac{1}{2}} - \frac{(2\pi)^{\frac{1}{2}}}{4\pi i} \int_{-i\infty}^{i\infty} \frac{e^{-m(s)}}{s^{\frac{1}{2}}(1 + s^2)^{\frac{1}{4}}} e^{xs/2} ds. \tag{28}$$

In the latter integral the path of integration can be deformed into a loop consisting of the two sides of the branch cut along the negative real axis. Then by use of the limit values

$$m(-\sigma \pm i0) = -m(\sigma) \pm i \arctan \sigma, \qquad \sigma > 0, \tag{29}$$

the solution for $f(x)$ becomes

$$f(x) = x^{-\frac{1}{2}} - \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \frac{e^{m(\sigma)}}{\sigma^{\frac{1}{2}}(1 + \sigma^2)^{\frac{3}{4}}} e^{-x\sigma/2} d\sigma, \tag{30}$$

in exact agreement with the result in [2, eq. (5.6)]. It does not seem possible to further evaluate the integral in (30).

Asymptotic expansions of $f(x)$ for small and large x may be derived directly from $\mathscr{L}\{f(x)\}$ expanded for $s \to \infty$ and $s \to 0$, respectively, by Abelian asymptotics of the inverse Laplace transform; see e.g. Doetsch [5, Kap. 7, 8]. It is found that the expansions are identical to those presented in [2, Sec. 6, 7] except for an error in the coefficient C_2 [2, eq. (6.2b)] which should be corrected by

$$C_2 = \frac{3}{2\pi^2}[-(\log 2)^2 + 2(\tfrac{5}{3} - \gamma)\log 2 - \gamma^2 + \tfrac{10}{3}\gamma + \tfrac{5}{3} + \pi^2], \tag{31}$$

where γ denotes Euler's constant. As a side-result we also have

$$\int_0^\infty f(x)dx = 0, \tag{32}$$

obtained from (26) by letting $\sigma \to 0$.

Finally we shall determine $F(z)$, as introduced in (3), by inversion of $G(s)$. From the definitions (16), (27), it easily follows that

$$m^*(s) = m(s) \pm i \arctan s, \qquad Im\, s \gtrless 0. \tag{33}$$

Accordingly the solution (25) for $G(s)$ can be rewritten as

$$G(s) = G^{(\pm)}(s) = \pm(2\pi)^{\frac{1}{2}} s^{-\frac{3}{2}} \left[1 - \frac{1 \mp is}{(1+s^2)^{\frac{1}{4}}} e^{-m(s)}\right], \qquad Im\, s \gtrless 0. \tag{34}$$

Both functions $G^{(\pm)}(s)$ are analytic in the complex s-plane cut along the negative real axis. In fact, $G^{(\pm)}(s)$ is the analytic continuation of $G(s)$ across the positive real axis starting from the half-plane $Im\, s \gtrless 0$. Now it can be shown that the inverse of the complex Laplace transform (7) is given by the formula

$$F(z) = \frac{1}{4\pi i} \int_{-\infty e^{i\beta}}^{\infty e^{i\beta}} \tilde{G}(s) e^{zs/2} ds, \tag{35}$$

where $\beta = \frac{1}{2}\pi - \arg z$, and $\tilde{G}(s) = G(s)$ when $Re\, z < 0$, while $\tilde{G}(s) = G^{(\mp)}(s)$ when $Re\, z > 0$, $Im\, z \gtrless 0$. If $Re\, z < 0$ ($Re\, z > 0$) the path of integration can be deformed into a loop around the branch cut along the positive (negative) real axis. Thus we find, by use of the limit values (18), (29),

$$F(z) = \frac{i}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \sigma^{-\frac{3}{2}} \left[1 - \frac{e^{-m(\sigma)}}{(1+\sigma^2)^{\frac{1}{4}}}\right] e^{z\sigma/2} d\sigma, \qquad Re\, z < 0, \tag{36a}$$

$$F(z) = \pm \frac{1}{(2\pi)^{\frac{1}{2}}} \int_0^\infty \sigma^{-\frac{3}{2}} \left[1 - \frac{1 \mp i\sigma}{(1+\sigma^2)^{\frac{1}{4}}} e^{m(\sigma)}\right] e^{-z\sigma/2} d\sigma,$$

$$Re\, z > 0, \quad Im\, z \gtrless 0. \tag{36b}$$

It is easily verified that the expressions (36a, b) yield the same value of $F(z)$ when $Re\, z = 0$. It is remarked that the function ϕ_1 appearing in [1, Sec. 4], is related to $F(z)$ through

$$\phi_1(\lambda, \mu) = A\, Im\, F((\lambda + i\mu)^2) \tag{37}$$

with $A = 0.755$.

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